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The Dynamics of Interacting Swarms

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14. ABSTRACT Swarms are self-organized dynamical coupled agents which evolve from simple rules of communication. They are ubiquitous in nature, and becoming more prominent in defense application. Here we report on a preliminary study of swarm collisions for a swarm model in which each agent is self-propelling but globally communicates with other agents. We generalize previous models by investigating the interacting dynamics when delay is introduced to the communicating agents. One of our major findings is that interacting swarms are far less likely to flock cohesively if they are coupled with delay. In addition, parameter ranges based on coupling strength, incidence angle of collision, and delay change dramatically for other swarm interactions which result in flocking, milling, and scattering.					
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1 Introduction

The emergence of complex dynamical behaviors from simple local interaction rules between pairs of agents in a group is a widespread phenomenon over a range of application domains, and results in coherent behavior called a swarm. Many striking examples can be found in biological systems, from the microscopic (ex., aggregates of bacterial cells or the collective motion of skin cells in wound healing) [4, 32, 17] to large-scale aggregates of fish, birds, and even humans [40, 14, 18]. These systems are particularly interesting to the robotics community because they allow simple individual agents to achieve complex tasks in ways that are scalable, extensible, and robust to failures of individual agents. In addition, these aggregate behaviors are able to form and persist in spite of complicating factors such as communication delay and restrictions on the number of neighbors each agent is able to interact with, heterogeneity in agent dynamics, and environmental noise. These factors define how swarm behaviors are effected by changes due to internal and external actors, allow the swarm to change its patterns and function [36, 35, 37, 15], as well as break apart and reform [29].

A number of studies show that even with simple interaction protocols, swarms of agents are able to converge to organized, coherent behaviors. Existing literature on the subject provides a wide selection of both agent-based [14, 18, 43, 40] and continuum models [10, 39, 32]. One of the earliest agent-based models of swarming is Reynolds’s *boids* [33], which simulates the motion of a group of flocking birds. The boids follow three simple rules: collision avoidance, alignment with neighbors, and attraction to neighbors. Since the publication of Reynolds’s paper, many models based on “zones” of attraction, repulsion, and/or alignment have been used as a means of realistically modeling swarming behaviors [26, 38, 44]. Systematic numerical studies of discrete flocking based on alignment with nearest neighbors we carried out by Vicsek *et al.* [42]. Stochastic interactions between agents are modeled in [31]. In recent years, improved computer vision algorithms have allowed researchers to record and analyze the motions of individual agents in biological flocks, and formulating more accurate, empirical models for collective motion strategies of flocking species including birds and fish [2, 16, 5].

Despite the multitude of available models, how group motion properties emerge from individual agent behaviors is still an active area of research. For example, [45] presents a simulation-based analysis of the different kinds of motion in a fish-schooling model; the authors map phase transitions between different aggregate behaviors as a function of group size and maximum number of neighbors that influence the motion of each “fish”. In [18], the authors use simulation to study transitions in aggregate motions of prey in response to a predator attack.

Interaction delay is a ubiquitous problem in both naturally-occurring and artificial systems, including blood cell production and coordinated flight of bats [22, 3, 28, 12, 11]. Communication delay can cause emergence of new collective motion patterns and lead to noise-induced switching between bistable patterns [23, 24, 20]; this, in turn, can lead to instability in robotic swarming systems

[44, 21]. Understanding the effects of delay and latency is key to understanding many swarm behaviors and functions in natural, as well as engineered, systems [37]. As another controlling factor of swarm function, network communication topology maybe used to improve and generalize swarm function. For example in [15], it was shown how to take known basic swarm modes, such as flocking, milling and rotating states, and use network topology to create new hybrid states that form new functions, such as shielding important swarm controlling agents.

In addition, many models make the mathematically simple but physically implausible assumption that swarms are globally coupled (that is, each agent is influenced by the motion of all other agents in the swarm) [30, 8, 7, 18, 41]. Global coupling is easier to analyze and a reasonable assumption in cases of high-bandwidth communication, with a sufficiently small number of agents. In contrast, we are interested in the collective motion patterns that emerge when global communication cannot be achieved. New behaviors can unexpectedly emerge when the communication structure of a network is altered, as in [46], where the stability of solutions for compromise dynamics over an Erdős-Renyi communication network is considered. However, in our system, we show robustness of emergent motion patterns to loss of communication links in presence of delayed coupling [37].

A third effect, which we do not consider here, is agent heterogeneity. Most existing work assumes that the members of the swarm are identical. However, many practical applications involve swarms that are composed of agents with differing dynamical properties from the onset, or that become different over time due to malfunction or aging. Swarm heterogeneity leads to interesting new collective dynamics such as spontaneous segregation of the various populations within the swarm; it also has the potential to erode swarm cohesion. In biology, for example, it has been shown that sorting behavior of different cell types during the development of an organism can be achieved simply by introducing heterogeneity in inter-cell adhesion properties [34, 13]. In robotic systems, allowing for heterogeneity in dynamical behaviors of swarm agents gives greater flexibility in system design, and is therefore desirable not only from a theoretical but also from a practical point of view [15].

In this preliminary work we are interested in swarms that are interacting. That is, the entire swarm is split into two parts initially, and then the two components approach each other as flocks. We then are interested in understanding the results based on physical parameters, such as attraction/repulsion length scales, delayed communication, and coupling strength. The model for the swarm we use is based on the the employed in [9], which describe a mathematically swarm model using the Morse potential. Recently, the authors in [1] studied the effects of swarm collisions using the Morse potential as a function of incident angle of the interacting flocks. However, in any robotic swarm there must be some delay between sensing other agents, and control actuation. We investigated the result of applying delay to the coupling terms of a swarm model. As a result, every agent behaves as though the other agents are at the positions they were at τ time units before. We found that the flocking state is unlikely

following collisions of delay-coupled swarms, even for fairly small values of τ . The implication is that it should be possible to have one swarm modify the intent of another by either capturing the swarm, or redirecting the mean direction of the flock.

2 The Basic Swarm Model

The swarm model we used for this preliminary study follows the model in [9]. It is a system of N agents in n -dimensional space with position vectors $x_i \in \mathbb{R}^n$, acting under the following equation of motion:

$$\ddot{x}_i = (\alpha - \beta|\dot{x}_i|^2)\dot{x}_i - \frac{\lambda}{N} \sum_{j=1, i \neq j}^N \nabla_{x_i} U(x_i, x_j), \quad (1)$$

where α, β, λ are constants, $U : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is a potential function of the two agents' position, and ∇_{x_i} represents taking the gradient with respect to x_i . Many potential functions can be used, as in [27], but a useful choice is the scaled Morse potential:

$$U(x_i, x_j) = C \exp(-|x_i - x_j|/l) - \exp(-|x_i - x_j|). \quad (2)$$

In particular, the parameter l can be used to set a ratio of repulsive and attractive length scales.

Following [1], we fix the values of α , β , C , and l . We also constrain our swarm to the plane. We choose $\alpha = 1$, $\beta = 5$, $C = \frac{10}{9}$, $l = \frac{3}{4}$. This means that, at long ranges, $\nabla_{x_i} U(x_i, x_j)$ will be attractive, and at short ranges it will be repulsive (see Fig. 1). Furthermore, the parameters choice places the swarm in the region which is defined as "catastrophic" [9], meaning that increasing the number of agents decreases the space between agents, so that the total size of the swarm approaches a limit in space.

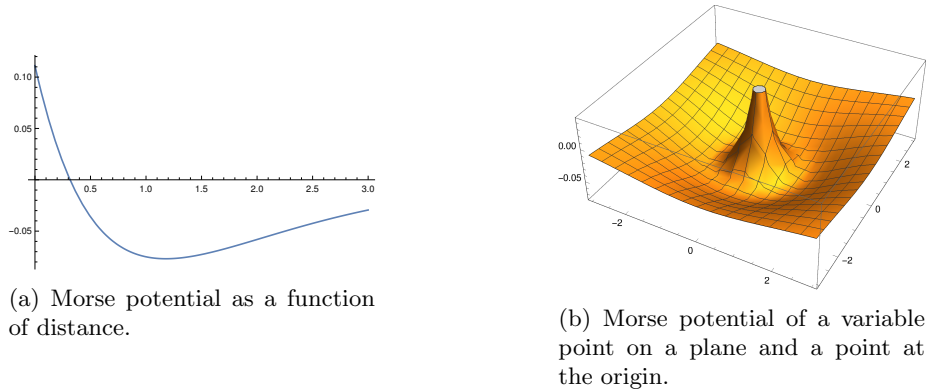


Figure 1: The Morse potential with $C = \frac{10}{9}$ and $l = \frac{3}{4}$.

When $\nabla_{x_i} U(x_i, x_j) \equiv 0$ for all $i, j \leq N$ with $i \neq j$, the dynamics is based solely on self-propulsion, and the resulting configuration of agents is called a flock [19]. That is, all of the agents when started in parallel, will translate with the same asymptotic velocity and direction. In the attraction-repulsion model, a flock undergoes translating motion at a constant speed $\sqrt{\alpha/\beta}$, and can withstand sufficiently small perturbations [6]. However, due to translation symmetry, there will always be one eigenvalue in the linearization about the flocking state equal to zero, which means the stability is neutral at best. Hence small perturbations may destabilize the flocking dynamics [25].

Since we are interested in flock collisions, two separate flocks are initialized and pointed towards each other with some incident angle, θ . We are interested in the behavior after collision, and how it varies for varying λ , θ , and τ .

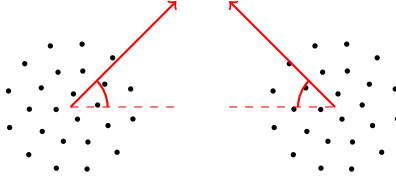


Figure 2: The incident angle of the collision, θ , which we vary as a parameter.

Flocking is not the only behavior that swarms display. They also can perform milling behavior, where the agents rotate around a stationary center of mass. To differentiate between the two behaviors, we calculate the polarization of the swarm as the directed sum of velocities. Specifically, for a system of N agents with position vectors x_i , the polarization P of the flock is defined as:

$$P(x_1, x_2, \dots, x_N) = \left| \frac{\sum_{i=1}^N \dot{x}_i}{\sum_{i=1}^N |\dot{x}_i|} \right|$$

Two limiting behaviors can be immediately seen to be reflected in the polarization. If the velocities are coherent, we expect a value of $P \approx 1$. If the velocities point in opposite directions, we expect $P \approx 0$. Therefore, polarization is a good measure of whether the system displays flocking ($P \approx 1$), or milling ($P \approx 0$). Additionally, two colliding swarms can scatter, in which they do not form a coherent group which either flocks or mills. In this case, we expect P to be neither close to 0 nor close to 1. See Fig. 3 for examples of the three types of swarm interactions.

3 Delay-Coupled Swarms

As mentioned above, since robotic systems possess latency in both communication and control actuation, we generalized the model to include a time delay τ in the interaction term, creating the following delay differential equation from equation (1):

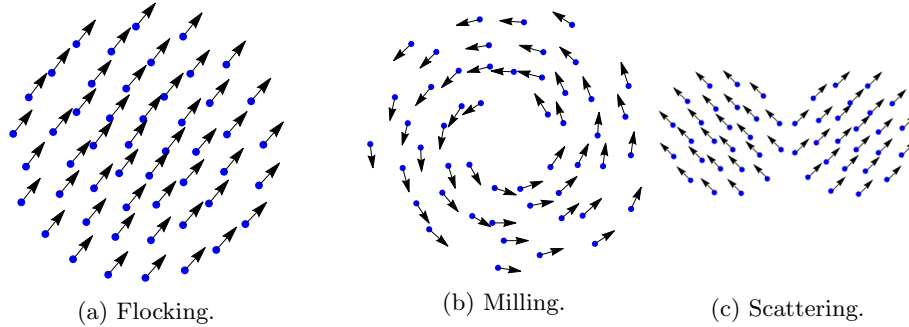


Figure 3: Flocking, milling, and scattering behavior in a Morse swarm.

$$\ddot{x}_i(t) = (\alpha - \beta|\dot{x}_i(t)|^2)\dot{x}_i(t) - \frac{\lambda}{N} \sum_{j=1, i \neq j}^N \nabla_{x_i} U(x_i(t), x_j(t - \tau)), \quad (3)$$

with U defined in Eq. (2). Since this is a delay differential equation, it requires an initial history function of position and velocity defined over $[-\tau, 0]$, for which we used translating motion of the two flocks along linear trajectories. We simulated two flocks of 25 agents, each swarm on a collision path with the other. We varied the values of λ , the coupling strength, and θ , the incident angle of the collision.

As a benchmark, we first simulated the results by performing the experiment without delay; i.e., $\tau = 0$, corresponding to a system with instantaneous interactions. We simulated 400 collisions and recorded the polarization after transients were removed, about 100 time units. We generated the plot in Fig. 4 showing how polarization can be used to quantify the resulting dynamics after the swarms collide.

From Fig. 4, we immediately observe several large scale features in parameter space. For low values of λ , scattering is likely unless the swarm motion is almost parallel in its initial state. For intermediate values of λ , flocking is likely to be observed, while for high values of λ , milling dominates the parameter region. A region of metastability exists between the flocking and milling states. The black lines are used to guide the eye to show the boundaries separating the final swarm states.

To see how the milling comes into play for the delay-coupled swarm, we start the flocks approaching each other at zero incidence angle. Figure 5 illustrates the progression from the initial flocks to the final milling swarm. It shows the potential of how one might use a defensive swarm to stop another by converting its flocking state, which is a traveling center of mass, to a milling state with a center of mass that is stationary.

In exploring the effects of delay on the parameter basins of swarm final states, we computed the polarization for three distinct values of delay: $\tau = 0, 0.1$, and 0.2 . The results of the average polarization plots using the above procedure are in Fig. 6.

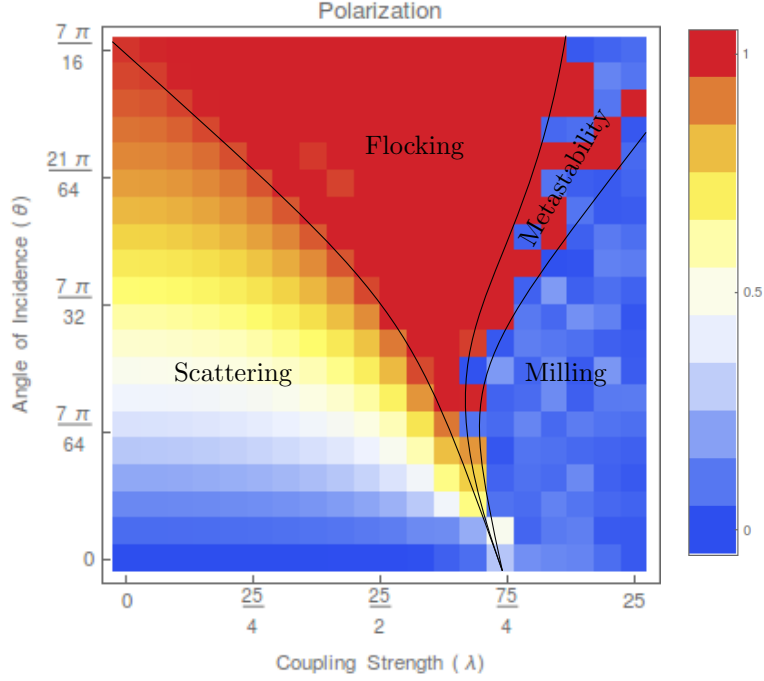


Figure 4: Polarization as a function of λ and θ . Note the distinct regions of flocking, milling, scattering, and metastability.

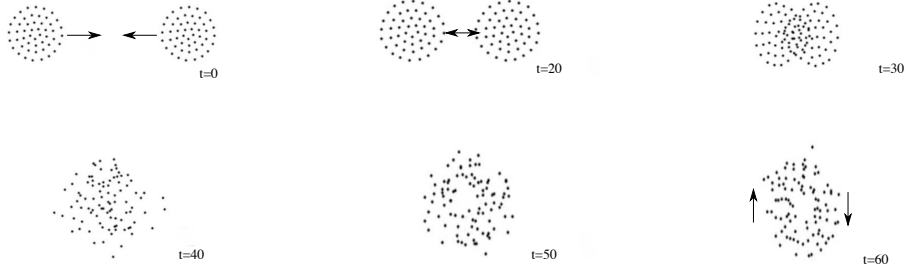


Figure 5: Time snapshots of swarm capture to a milling state. Delay here is $\tau = 0.1$, and other parameters were chosen in the milling regime of Fig.

We can note several things about the introduction of delay in panels of Fig. 6. Reference from the $\tau = 0$ case, one see a gradually shrinking of the flocking region, with an increase in milling parameters. Even at the relatively small delay of $\tau = 0.2$, flocking behavior nearly vanishes. Instead, scattering behavior becomes more likely for low λ , and milling behavior becomes more likely for high λ .

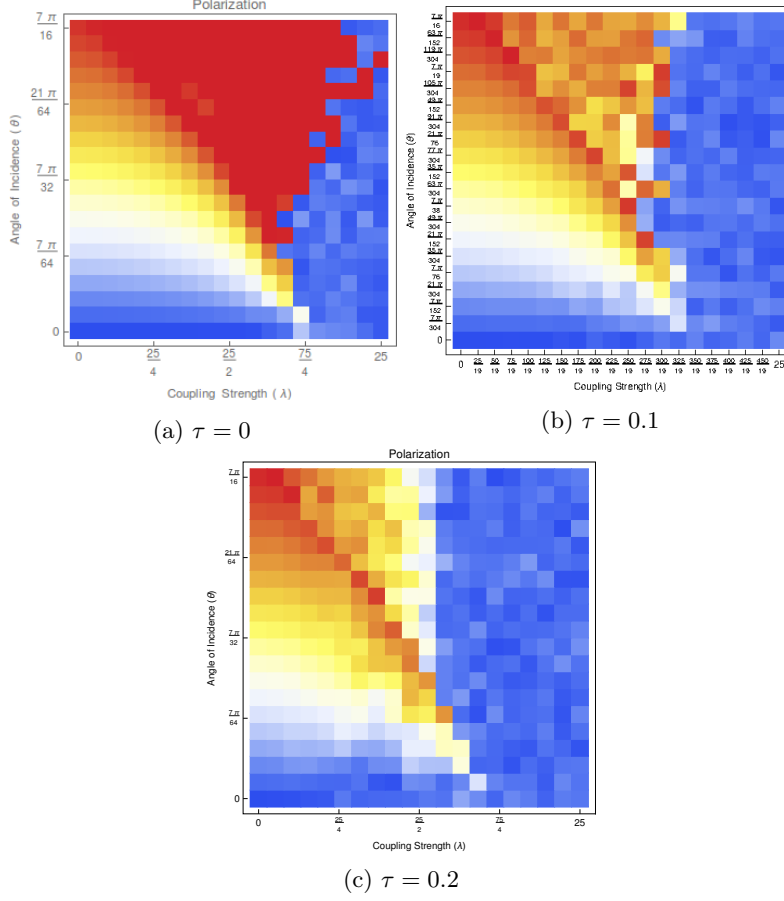


Figure 6: Polarization as a function of λ and θ for 3 distinct values of τ .

To check the overall dependence of swarm final state on both coupling strength and delay, we fixed the incidence angle θ at $\frac{\pi}{4}$, and varied τ finely, to observe polarization as a function of τ and λ . The results are shown in Fig. 7. This confirms that as delay increases steadily, milling behavior largely replaces flocking behavior.

4 Conclusion

The preliminary results of this work reveal certain features about how interacting swarms upon collision modify their behavior. One major finding is that the existence of a swarming flocking state depends strongly on the delayed interaction between agents. If delays can be induced into the communication network of a flocking swarm, the results show that even small delay disrupts the motion

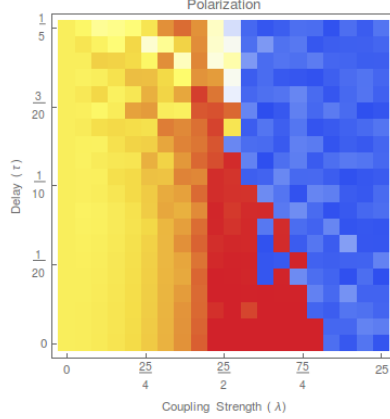


Figure 7: A plot of polarization as a function of λ and τ for fixed θ . Note that flocking behavior all but disappears above $\tau \approx 0.1$.

of the center of mass, thus disabling the flocking state.

In the case of globally coupled interacting swarms which are equal in number, the resulting swarm states may be in one of three states: Flocking, Milling, and Scattered. However, the results from this study depend heavily on the initial states, such as velocity and incidence angle, in addition to delay magnitude. Nonetheless, the final states of colliding swarms may result in modifying the center of mass velocity, change the navigation course of the swarm from its original path, and capture another swarm. The potential of using one swarm to modify another also depends on how it is sensing other agents, which could be through lidar, vision, or from a third party, such as a mother ship equipped with radar.

The sensitivity of the flocking state also has some other consequences for modeling. For example, it shows that self-organizing real flocking biological systems which may be modeled by swarms may have more complicated underlying dynamics and adaptive controls than are not included in the deterministic equations of motion. There may be some other factors not modeled here which biological flocks may use to compensate for actuation delays, noise, and communication latency. In particular, since the linear stability of the flocking state is neutral, there must be other individual agent controls to make the flocking more robust to perturbations.

The preliminary results here also show that globally coupled swarms with delayed communication cannot be accurately modeled by an instantaneous model. Similar to work shown in [37], delays cause new states to emerge that are not observed in systems with instantaneous communication. Moreover, it is also possible to create new when delay is included in the coupling terms along with changes in network communication topology which is not global, as shown in [15]. In the case of a given Morse potential here, simulations accurately modeling delay-coupled swarms must take the delay into account to get accurate

results.

Finally, since only the Morse potential was used here to model interacting swarm dynamics, other communication and coupling schemes need to be studied to see what kinds of general statements may be made, and which conclusion pertain only to specific swarm models. The end results of such research should prove to open new areas of interacting and combating swarms in the future.

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